Distribution of stress in a craze of the tip of a uniformly extending crack

E. Passaglia
Center for Materials Science, National Bureau of Standards, Washington, D.C. 20234, USA
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A model of a craze at the tip of a uniformly extending crack is developed which permits the calculation of the stress distribution in the craze. In accord with experimental observations by Kramer [11] the craze is modelled as a collection of independent fibrils that draw from the substrate by a process akin to the drawing of textile fibres with necking. Except at the very tip of the craze where complex yielding type phenomena occur, the stress in the craze is taken to correspond to the drawing stress. The craze stress is treated as the cohesive crack closing stresses in the Barenblatt treatment of crack tips. The principal fact used in the development is that the drawing stress depends upon the rate of draw and hence upon the slope of the craze displacement. This leads to a non-linear integral equation for the craze stress. Using an empirical relation between drawing stresses and rate of draw, this equation is solved for the stress distribution in the craze by numerical methods. The distribution shows peaks at the craze tip and at the crack tip as observed in some experiments. The magnitude of the peaks depends upon the materials parameters used. For certain values of these parameters, the constant stress Dugdale model yields a good approximation to the displacement profile.

(Keywords: craze displacement profiles; craze stresses; fibril drawing; stress distribution)

INTRODUCTION

The Dugdale model[1] of a linear plastic zone at the tip of a crack has been used extensively and with considerable success as a representation for crazes at crack tips. The model has been used for both theoretical investigations[2,3] on the effect of viscoelasticity on crack extension and for the analysis of experimental data obtained using interference optics techniques for displacement profiles[4–8]. The observed craze displacement profiles can be reasonably represented by this model, although some differences occur[4,7,8]. The model has also often been used to calculate crack opening displacements which are then used to provide a criterion for crack growth.

The Dugdale model, developed for plane stress yielding in this steel sheets, assumes that the stress in the yielded zone at the crack tip is a constant independent of position and equal to the yield stress of the material. For crazes in thin films of polymers (which have nevertheless been shown to be representative of crazes in bulk polymer provided the film is not too thin[9,10]) the stress distribution in the craze can be determined with high resolution by electron microscopic observation of the craze displacement profiles[11]. This can be done both for crazes without cracks[12] and for crazes grown from holes to represent cracks in the thin films[11]. These latter experiments uniformly show that the stress within the craze is not uniform. Typically, there are peaks in the stress distribution at the tip of the craze and at the tip of the crack. While in many cases the peaks are not very great, in other cases they are significant[13]. We thus have the situation in which some experiments are consistent with the craze stress being constant over the craze, while other experiments show that it is not.

A possible method of analysis of the stress distribution in a craze is provided by the demonstration that craze thickening occurs by the process of fibril drawing from the undeformed substrate[14] by a process akin to drawing with necking[15] in a textile fibre, with the craze fibril corresponding to the drawn fibre. It has recently been shown that there is indeed a similarity between macroscopic and craze fibril drawing[16]. The drawing stress of a textile fibre depends upon the rate of draw, increasing as the draw rate increases, and, if the craze thickening process is indeed analogous to fibre drawing, the craze stress in a craze at the tip of an extending crack should depend upon the crack velocity. Moreover, since under these circumstances the rate of fibril drawing from the substrate will depend upon the slope of the craze displacement curve, the stress can only be constant along the craze if the slope of the displacement is constant. This cannot be the case, and hence the stress along the craze cannot be a constant. The magnitude of the stress variation is, however, a matter for calculation.

In addition to these purely rate dependent arguments, there is the matter of the yielding behaviour of polymers with necking that argues against a uniform craze stress. Upon yielding, the stress decreases to the drawing stress, a phenomenon commonly called 'strain softening'. While the processes that occur at the tip of a craze are far more complex than the yielding of a textile fibre, in the interpretation of craze thickening as a drawing process, the processes at the crack tip must somehow correspond to the yielding phenomenon and the stress would be expected to drop just behind the craze tip. This will be discussed more fully later, and again the magnitude of the effect is a subject for computation.

The purpose of this paper is to develop the concept of craze widening by fibril drawing into a method for determining the stress distribution in a craze at the tip of a uniformly extending crack. The craze is considered as a

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collection of essentially independent fibrils. The principal fact used is that the drawing stress depends upon the rate of draw, and hence upon the slope of the displacement. Taking reasonable physical account of these subtleties leads to a stress distribution that has peaks at the craze and crack tips. The magnitude of these peaks depends upon the material parameters associated with the drawing process.

THE MODEL

We consider a uniformly translating Mode I crack with craze at its tip. We assume that the length of the craze is a constant and very short compared to the crack length. The properties of the substrate and of the drawn craze fibrils are considered to be time-independent, i.e. we do not consider these to be viscoelastic materials. Such considerations would add greatly to the complexity of the problem. For our analysis we adopt the Barenblatt theory[17], as previously done for cracks advancing in viscoelastic materials[2] and for the effects of the viscoelasticity of crazes at the tip of non-moving cracks[18]. This model is only applicable to short crazes at the tips of long cracks, which is reasonable for the situation considered here. This model has another important result, In it, there is no stress singularity at the tip of the craze or at the tip of the crack. It should be noted here, that this treatment considers the stress in the craze to be the cohesive crack-closing stresses of the Barenblatt theory. In their absence, the crack would extend to the craze tip, where a stress singularity would exist. The cohesive (craze) stresses in the Barenblatt theory serve to cancel this singularity. The stress at the crack tip, which exists in the craze, is not specified. This is well reviewed by Schapery[2].

This removal of the singularity at the craze tip by the closing (craze) stresses has an important corollary. Barenblatt has shown that under these circumstances, the derivative of displacement at the tip of the craze is zero, i.e. the displacement there is cusp shaped. As will be seen, this has an important bearing on our analysis.

We consider a crack of length *2a* in an infinite material, and consider only one of its tips in *Figure 1*. The tip of the crack is advancing with uniform velocity v. The length of the craze is α. Two coordinate systems are specified, one denoted by x with origin located at the centre of the crack, and one with origin at the craze tip, denoted by ξ and ζ.

Now, under the assumed conditions of uniform translation, we have:

$$\partial w/\partial t)_x + v\partial w/\partial x)_t = 0 \tag{1}$$

where w is the craze displacement. This gives:

$$\partial w/\partial t)_x = v\partial w/\partial \xi)_t \tag{2}$$

From the Barenblatt theory, we can express w as a function of the stress within the craze (2)

$$w(\xi) = C_e/2\pi \int_0^\alpha \sigma(\zeta)\left\{2(\xi/\zeta)^{1/2} - \ln\left|\frac{\zeta^{1/\xi}+\xi^{1/2}}{\zeta^{1/2}-\xi^{1/2}}\right|\right\}\mathrm{d}\zeta \tag{3}$$

where *Ce* is the plane strain compliance of the uncrazed substrate and $\sigma(\zeta)$ is the craze stress. This equation may be differentiated to give:

$$\frac{\partial w}{\partial \xi} = \frac{C_e}{2\pi}\int_0^\alpha \sqrt{\frac{\xi}{\zeta}}\frac{\sigma(\zeta)\mathrm{d}\zeta}{\xi-\zeta} \tag{4}$$

Now we define new variables

$$\eta = \zeta/\alpha \qquad \beta = \xi/\alpha \tag{5}$$

and $\sigma(\eta)$, combine equation (4) with equation (2) to obtain:

$$\dot{w} \equiv \left(\frac{\partial w}{\partial t}\right)_x = v\left(\frac{\partial w}{\partial \xi}\right)_t = \frac{vC_e}{2\pi}\int_0^1 \sqrt{\frac{\beta}{\eta}}\frac{\sigma(\eta)\mathrm{d}\eta}{\beta\eta} \tag{6}$$

The integral exists as a Cauchy principal value provided $\sigma(\eta)$ is continuous.

This is the fundamental equation for our analysis. It says that the time rate of change of displacement at a given position is given by an integral of the stress over all points in the craze. However, by hypothesis, the stress at the given point depends upon the rate of draw:

$$\sigma = \sigma(\dot{w}) \tag{7}$$

i.e. the process is essentially a non-linear viscous deformation. Hence we are given a type of integral equation for the stress within the craze. Before proceeding to discuss the solution of this equation we discuss the situation at the craze tip, for it presents some subtleties.

Situation at the craze tip

The situation at the craze tip is very complex. The craze grows by a Taylor meniscus instability[11,19]. For our purpose two interrelated questions need to be answered. The first is, 'When are the craze fibrils fully formed so that they may be considered to be drawing?', and the second, which is related to the first is, 'At what point may the stresses in yielded-crazed regions be considered to be closing stresses in the Barenblatt sense?'

We will attempt to answer these questions by pressing the analogy between craze thickening and fibre drawing. This is a dangerous procedure right at the craze tip for

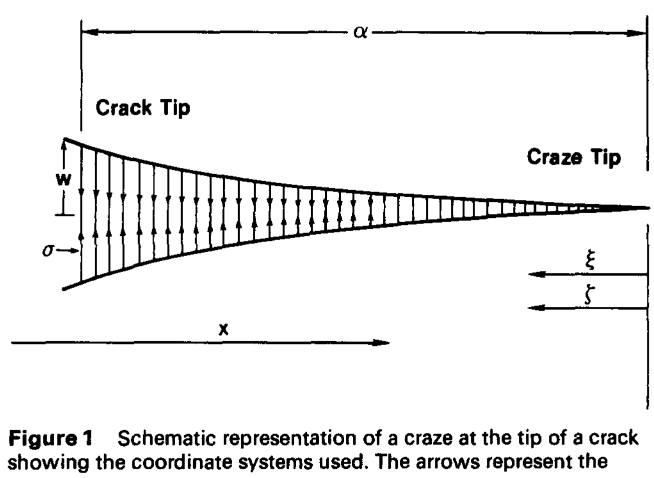

Figure 1 Schematic representation of a craze at the tip of a crack showing the coordinate systems used. The arrows represent the craze stress σ

there the analogy might very well break down. In fibre drawing, the stress rises to a yield point at a strain of several per cent[15,20]. At this point, the tangent to the stress–strain curve is parallel to strain axis, and this is a point of instability. Necking does not, however, occur at this point. It begins at a somewhat higher strain, and what visually appears to be a fully formed neck appears at a still higher strain, at which point the stress is lower than the yield stress but higher than the drawing stress. The stress then continues to decrease until it achieves the value of the drawing stress, at which point the new fibre diameter is established. The engineering stress then remains sensibly constant until all the material is drawn.*

We hypothesize that in broad terms this process can be taken over to the craze tip problem. Thus we assume that yielding will occur at the craze tip, when the stress achieves a value corresponding to the yield stress, even though no craze fibrils may be observable at that point. That is, in answer to our second question, we state, 'when the material becomes highly non-linear, even though craze fibrils may not be yet formed'. We shall then assume that very shortly thereafter, craze fibrils are formed, and the drawing process proceeds. Where this is in the Taylor meniscus region remains a moot point. From the results of calculations to be shown later, it would appear to be in the region of the meniscus fingers. In any event, we assume that the stress at the tip of the craze corresponds to the yield stress, and fibril formation occurs very shortly thereafter.

It is clear from equation (6) that provided the stress is finite, the rate of change of displacement is zero at the craze tip. The stress at that point cannot, therefore, be caused by the drawing mechanism, for $\partial w/\partial t)_x$ will always be zero at the craze tip. This is the condition that no singularity exist at the craze tip in the Barenblatt formulation. We are therefore forced to assume that at the tip of the craze, the stress corresponds to the yield stress, and that this stress, unlike that at all other points within the craze after fibrils are formed, is not determined by the slope of the displacement. It will, of course, be determined by the velocity of the crack, for the yield stress is also dependent upon the rate of deformation, but not by the slope of the displacement curve.

Situation at the crack tip

From equation (6) it is seen that the crack tip ($\beta = 1$), $\dot{w}$ becomes infinite if $\sigma(\eta)$ is finite at that point. If $\sigma(\eta)$ approaches zero linearly or faster, $\dot{w}(1)$ becomes finite, since $\sigma(\eta)$ is always positive. These considerations will become important when a solution of equation (6) is attempted. To do this we need to consider the time dependence of the drawing process.

Drawing process

There have been numerous studies of the rate dependence of the yielding process, but relatively fewer on the drawing process[15,20–27]. It is important to note that the whole stress–strain curve is dependent upon the rate of draw. Indeed, Matsuoka[20] has shown that if $\sigma_0(\dot{\varepsilon})$ is the stress–strain curve obtained at a strain rate of $\dot{\varepsilon}_0$, then the stress–strain curve at a new strain rate may be approximately obtained from the relation:

$$\sigma(\varepsilon) = \sigma_0(\varepsilon)(\dot{\varepsilon}/\dot{\varepsilon}_0)^n \tag{8}$$

where n is a constant of the order of 0.025–0.08 for many polymers. This constituitive equation has recently been used by Kramer and Hart[28] in a theoretical treatment of crack growth in polymers. Equation (8) is somewhat different from other results that show that the strain rate dependence of the yield stress is somewhat different from that of the draw stress[15]. In these other studies, at a constant temperature, both the yield stress and draw stress are linear on a plot of stress vs. the logarithm of the strain rate, i.e.

$$\sigma = \sigma_r + \sigma_s \log(\dot{\varepsilon}/\dot{\varepsilon}_r) \tag{9}$$

where σ_r is the yield or drawing stress at the reference strain rate $\dot{\varepsilon}_r$. The results of Matsuaka imply that the rate coefficient σ_s is the same for all portions of the stress–strain curve, and indeed it is not greatly different for the yield stress and the draw stress. We shall adopt equation (9) as giving the rate dependence of the stress–strain curve for all portions beyond which a neck has formed. From the previous discussion, this occurs at a strain somewhat beyond the yield strain but smaller than the strain at which the stress has achieved the constant value which is the draw stress. The differences between using equations (8) and (9) are negligible for our purposes.

For our purposes it is best to write equation (8) in the form:

$$\sigma = \sigma_r + m\sigma_r \log(\dot{\varepsilon}/\dot{\varepsilon}_r) \tag{10}$$

where $m = \sigma_s/\sigma_r$. Investigation of the draw behaviour for a number of polymers shows that m varies from about 0.03 to 0.1[15,20–27] depending on the polymer, the temperature and, of course, slightly on the reference strain rate. In our calculations we shall treat m as a parameter and show calculations for various values of it.

One other point we need to discuss is the difference between the yield stress and the draw stress. Again, this varies and, depending on the polymer, temperature and strain rate, the ratio of the draw stress to the yield stress varies from about 0.55 to 0.9[15,20–27].

SOLUTION

Formulation of the final equation

The final formulation of our problem comes from a combination of equations (6) and (10). In order to do this we need a relationship between $\dot{\varepsilon}$ and $\dot{w}$. Since $\dot{\varepsilon}$ in equation (10) refers to engineering strain, we need a 'gauge length' to convert w to ε. This is conveniently provided by τ_0^m, the maximum value of the 'primordial craze' as defined by Lauterwasser and Kramer[12]. From this we have:

$$\dot{\varepsilon} = w/\tau_0^m \tag{11}$$

and, at a specified crack velocity:

$$\dot{\varepsilon}/\dot{\varepsilon}_r = \dot{w}/\dot{w}_r \tag{12}$$

In this equation, $\dot{w}_r$ is the time rate of change of displacement under reference conditions, which we are free to choose. Now, from equation (6), $\dot{w}$ depends upon position within the craze and the velocity with which the

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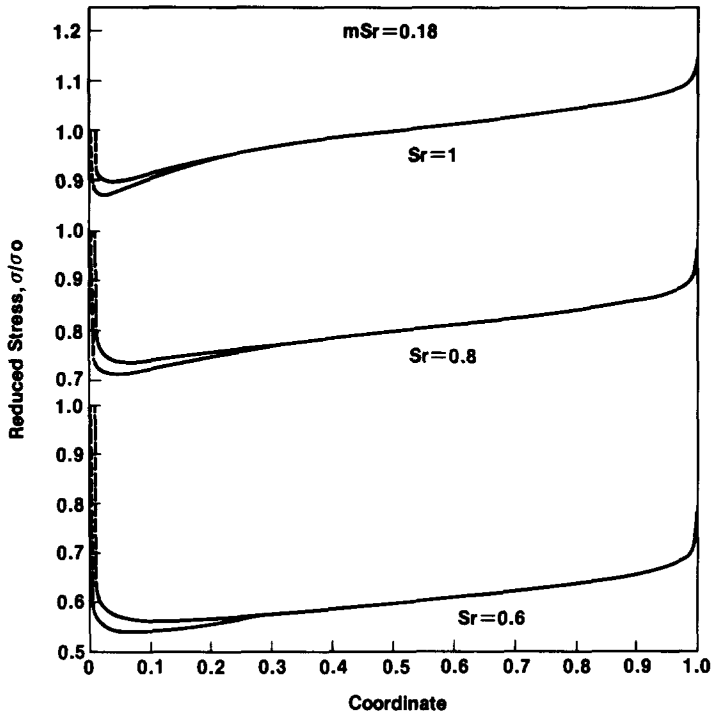

Figure 2 The ratio of the craze stress to the stress at the craze tip plotted as a function of position in the craze for a rate sensitivity value of 0.18 and for three values of the reduced mid-point stress. The craze tip is at the left. The two curves near the craze tip are for different ways of handling the stress at the craze tip as discussed in the Appendix

crack is moving. If we denote the integral in equation (6) by $I(\beta)$, we obtain:

$$\dot{w}_r = Ce\,v_r I(\beta_r)/2\pi \qquad (13)$$

and hence,

$$\dot{\varepsilon}/\dot{\varepsilon}_r = \dot{w}/\dot{w}_r = vI(\beta)/v_r I(\beta_r) \qquad (14)$$

If we were to compare cracks moving at different velocities, i.e. if the crack under consideration were progressing at a different velocity than our reference crack we would have a very difficult problem, for, as previously discussed, the yield stress would be different, and there is no reason to expect that the stress distributions in the craze would be the same under these circumstances. Hence, we consider our reference crack to be moving at the same velocity as the crack under consideration. We are still at liberty to choose a reference point within the craze where we wish. For this calculation we have chosen it at the mid point of the craze. Finally, combining equations (10) and (13) we obtain:

$$\sigma(\beta) = \sigma_r + m\sigma_r \log[I(\beta)/I(0.5)] \qquad (15)$$

where:

$$I(\beta) = \int_0^1 \sqrt{\frac{\beta}{\eta}}\,\frac{\sigma(\eta)\mathrm{d}\eta}{\beta - \eta} \qquad (16)$$

For purposes of calculation, we divide both sides of this equation (15) by the yield stress σ_y to obtain:

$$S(\beta) = S_r + mS_r \log I(\beta)/I(0.5) \qquad (17)$$

where $S(\beta) = \sigma(\beta)\sigma_y$ and $S_r = \sigma_r/\sigma_y$. It is clear from the way we have formulated this problem that at the mid point of the craze, $S(\beta) = S_r$. It is also clear that S_r is related (but not necessarily equal) to the ratio of the drawing stress to the yield stress. Clearly, we have two parameters at our disposal, S_r and m, and we carry out calculations for various values of these two. At the craze tip, $S(0)$ is always equal to unity.

Solution of the final equation

Equation (17) is an integral equation which needs to be solved for the stress distribution $S(\beta)$. It is different from standard types of integral equations in that the unknown function is equal to a function of an integral (the log in this case). It cannot be solved therefore by standard techniques. We consequently have had to solve this equation numerically by successive approximation. The method of solution is given in the Appendix.

RESULTS AND DISCUSSION

Stress distributions

Stress distributions were calculated by the methods given in the Appendix for several values of the parameters. These were as follows. The values of the rate coefficient represented by mS_r were chosen to be 0.06 and 0.18. These were chosen to represent polymers with moderate to low and high drawing rate sensitivity, respectively. The values of the reference stress, S_r, which represents the ratio of the craze mid-point stress to the yield stress were chosen to be 1, 0.8 and 0.6. The first of these is patently illustrative and designed to illustrate qualitatively what happens to a Dugdale model where draw rate sensitivity is added. The last was chosen because recent approximate calculations by Kramer[29] show that the ratio of the craze drawing stress to the craze initiation step is about 0.5. Finally, the value of 0.8 was chosen as an intermediate value.

The results of these stress distribution calculations are shown in *Figures 2* and *3*. *Figure 2* gives the results for

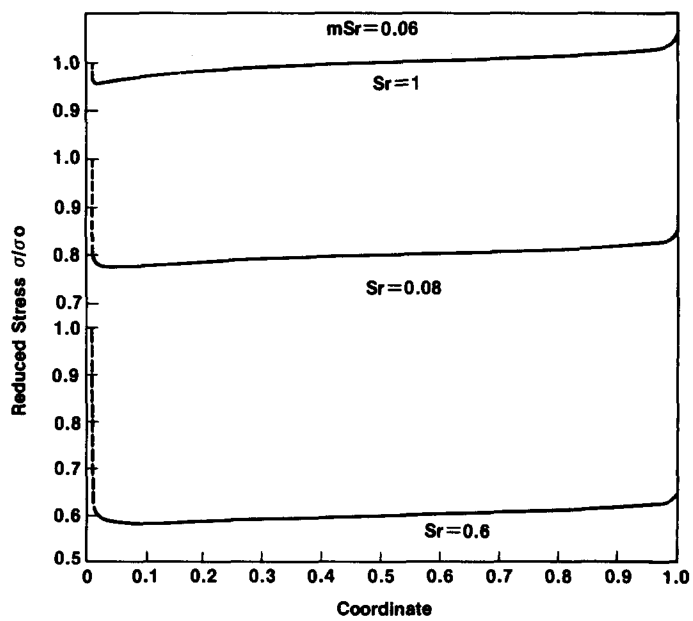

Figure 3 Same as *Figure 2* but for a rate sensitivity of 0.06

$mS_r = 0.18$, and *Figure 3* gives them for $mSr = 0.06$. In *Figure 2*, two results are given for each value of Sr. The upper curve in each case is the result when the stress was fixed at the yield stress at the origin ($\eta = 0$) and at $\eta = 0.01$ as discussed in the Appendix. The lower curve represents the case where it was fixed only at the origin. The results for this second case for $mSr = 0.06$ (*Figure 3*) are not sufficiently different from the results shown to make their presentation worthwhile, hence they are not shown. In each Figure, the dotted lines represent those stress regions that did not enter into the solution iteration as discussed in the Appendix.

In each case the curves are characterized by a peak in the region of the craze tip and one in the region of the crack tip. Part of the peak at the origin arises from the stipulation of the stress behaviour at the origin, but part of it arises from the solution of equation (17). In all cases, the stress drops to a value below S_r, rises gradually at a rate dependent on both mS_r and S_r, and then rises abruptly at the crack tip. We will now discuss the behaviour at the craze tip and at the crack tip.

The behaviour at the craze tip represents 'strain softening', and the magnitude of this softening is roughly represented by Sr. However, even where Sr is unity, strain softening occurs. A qualitative investigation of the form that $S(\eta)$ must take in order to achieve a solution of equation (17) shows that a peak in the stress distribution at the origin is necessary. Without such a peak, the calculated stress distribution falls to zero much too slowly as the origin is approached to permit a solution. Bearing in mind that by the reasoning described above the stress at the origin cannot be zero and must represent the yield stress, the solution to equation (17) appears to demand that strain softening occur. This is an unexpected but reasonable result.

The behaviour at the crack tip arises for a different reason. Inspection of equation (6) shows that for finite stress at the crack tip ($\beta = 1$), $\dot{w}$ will always diverge and hence, from the empirical stress law, equation (9), so will the calculated stress. Moreover, if the assumed stress is permitted to drop to zero at the crack tip, $\dot{w}$ will always be finite, for σ elsewhere is always positive. Hence, the stress calculated from $\dot{w}$ cannot be zero in our model, and the only way to achieve a solution to equation (12) is for the stress to diverge at the crack tip. The divergence is very slow. Thus, for a stress that rises linearly the divergence of equation (6) is logarithmic, and the calculated stress diverges as the logarithm of this. The points shown in the graphs are calculated for a coordinate value of 0.999999.

In a recent theoretical treatment of crack growth velocity as a function of applied stress intensity factor, Kramer and Hart[28], were led to the conclusion that a classical $r^{-1/2}$ stress singularity exists at the crack tip. That singularity is much more rapid than the one found in this work. More importantly, their singularity is independent of material properties and arises because of a need for crack extension force, namely K, in their theoretical model of the crack growth process. Our singularity, on the other hand, arises because the rate dependence of the draw rate. However, since it is difficult to conceive of a crazing material with a stress that does not increase with the rate of draw, we conclude that in a moving crack with a craze at its tip, a stress singularity will tend to occur at the crack tip. We say, 'tend to occur' because equation (9) is only approximate and because failure will obviously occur before the stress approaches infinity. Thus, Kramer and Hart[28], based on the work of Trassaert and Schirrer[30], were led to conclude that the fibrils right at the crack tip failed by fibril creep. Since we have not addressed the mechanisms of crack advance in our work we should not pursue this discussion but it seems reasonable to conclude that failure occurs very close to what we have identified as the crack tip where the stress rises.

The curves we have calculated qualitatively have the same features as observed by Kramer and coworkers[11,13]. We cannot make too much of a comparison because of the great differences in the experimental situation. Ours is a highly idealized experimental situation of a uniformly advancing crack-craze, whereas the Kramer experiments are on static crazes in thin films. Nevertheless, in many cases the similarity is striking.

More pertinent for our purposes are experimental observations on crazes at crack tips carried out by interference microscopy. These have recently been reviewed by Doll[4]. In these experiments the form of the displacement curves is determined, and we now proceed to discuss this.

Displacements

It is clear from our formulation that the curves of stress vs. position are essentially curves of the log of $\partial\omega/\partial\xi)_t$. Hence, the displacement may easily be calculated from these curves. The pertinent equation is:

$$w(\beta) = Ce\alpha/2\pi \int_0^\beta I(\beta)\mathrm{d}\beta \tag{18}$$

where $I(\beta)$ is the integral appearing in equation (6) and α is the craze length. We have carried out this integration and the results are shown in *Figures 4–6*. In these Figures the curves have been normalized by dividing by half the crack opening displacement, i.e. $w(1)$. This amounts to plotting in units of

$$Ce\alpha/2\pi \int_0^1 I(\beta)\mathrm{d}\beta \tag{19}$$

or

$$\frac{K^2 C_e}{2\sigma(0)} \frac{\int_0^1 I(\beta)\mathrm{d}\beta}{\left[\int_0^1 \frac{S(\eta)\mathrm{d}\eta}{\eta^{1/2}}\right]^2} \tag{20}$$

where K is the stress intensity factor.

Figure 4 shows the calculated displacement curves for the cases where $mSr = 0.18$. This value was selected to illustrate the difference between these calculated curves and that for the constant stress Dugdale case which is also shown in this Figure. The curve for the Dugdale case cuts across the calculated curves. *Figures 5* and *6* show the situation at the origin at a higher resolution. *Figure 5* is for $mSr = 0.18$ and *Figure 6* is for $mSr = 0.06$. In both of these

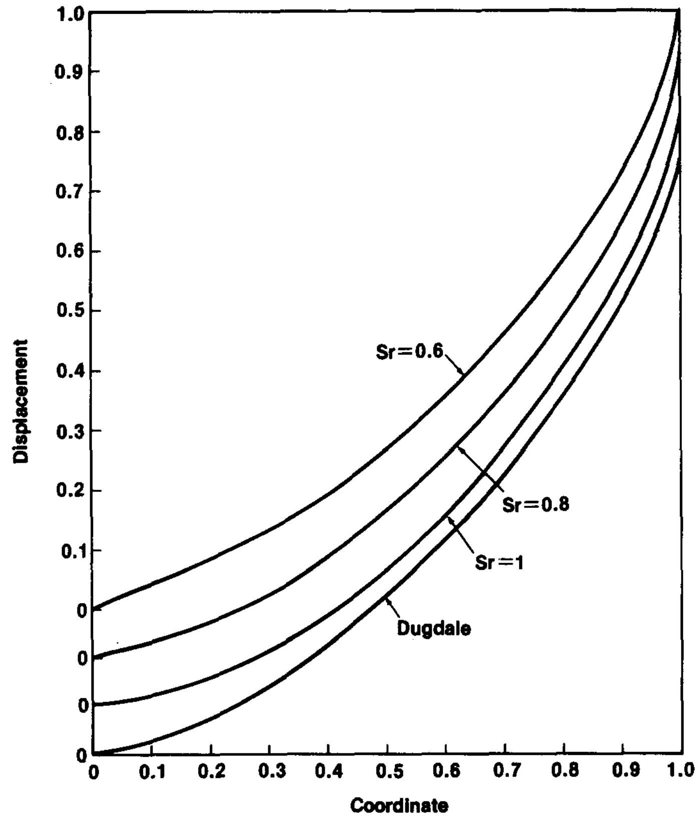

Figure 4 Displacements calculated for the stress distributions shown in *Figure 2*. Displacements for the Dugdale case have been added for comparison. The curves have been displaced upwards for clarity

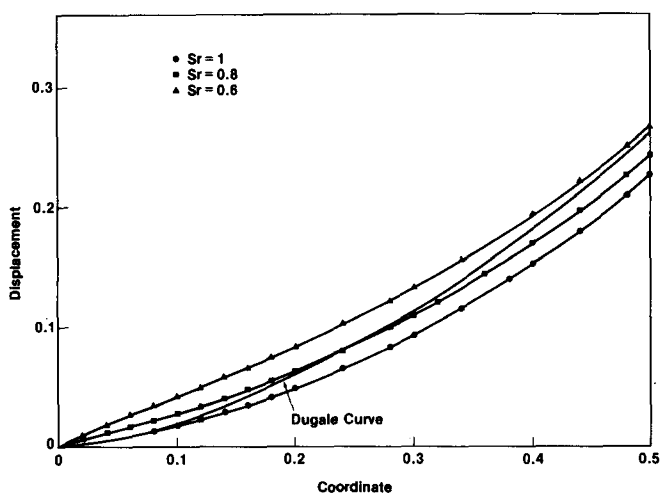

Figure 5 The portion near the craze tip of the curves in *Figure 4* illustrating the effect of the peak in the stress at the craze tip

Figures, the curves for $S_r=0.6$ and 0.8 show a noticeably steeper rise at the origin than does the Dugdale case, as is to be expected from the peak in the stress distribution calculated for these cases. It should be noted, however, that in *Figure 6*, the curve for $S_r=0.8$ in *Figure 6* is a reasonable approximation to the Dugdale curve. Indeed, the differences do not appear to be much larger than the differences sometimes observed between experimentally determined displacement profiles and those calculated for the Dugdale model[4,5,7]. Other choices of parameters could make these differences still smaller, but we have not pursued this. Suffice it to say that at least a portion of the

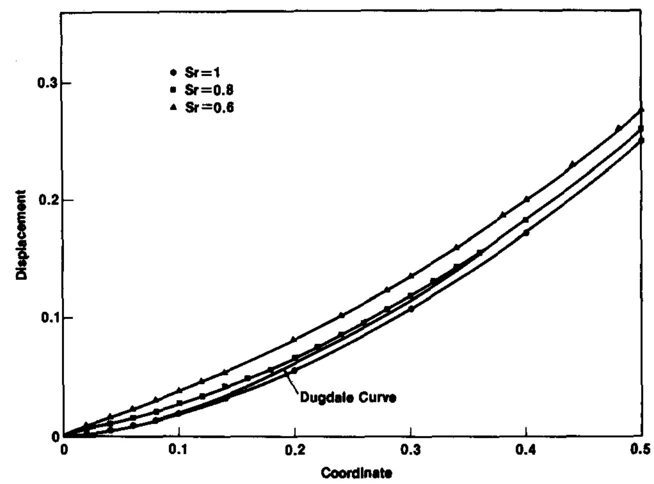

Figure 6 Same as *Figure 5* but for a rate sensitivity of 0.06. The associated stress distribution curves are in *Figure 3*

differences between the Dugdale model and the experimentally determined displacement profiles can be explained by the existence of peaks in the stress distribution as observed by Kramer *et al.* and as calculated in this paper. It is also clear that for materials with smaller rate coefficients than used here, the Dugdale model would provide an excellent approximation.

CONCLUSION

A model has been formulated for calculating the stress distribution in a craze at the tip of a uniformly advancing crack. The craze is modelled as a collection of independent fibrils, and thickens by drawing in a process analogous to necking and drawing in a textile fibre. At the tip of the craze yielding occurs, and shortly thereafter the craze develops. The stress in the craze is determined by the drawing stress. Using the fact that the drawing stress increases with rate of draw, a non-linear integral equation for the craze stress is formulated. Using an empirical equation for the dependence of stress on draw rate, this equation is solved for the stress distribution. This distribution shows peaks in the stress at both the craze tip and at the crack tip as observed experimentally[11]. The former arises from strain softening, and this appears to be demanded by the model. The latter arises from the behaviour of the displacement at the crack tip and is also demanded by the model. It is suggested that the presence of these peaks account for at least some of the differences observed between measurements of displacement profiles in bulk specimens and those calculated from the Dugdale model[4].

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APPENDIX

The numerical solution of equation (17) will be discussed in three sections: protocol, integration and treatment at craze and crack tips.

A1. Protocol

The protocol that was used is as follows:

(a) Assume an initial trial solution $S_0^a(\eta)$ at the source points η for the stress distribution. The conditions for this choice are described in Section A3.

(b) Choose values of S_r and mS_r.

(c) Compute $I(\beta)/I(0.5)$ from the assumed $S_0^a(\eta)$, and from this a calculated value of the stress $S_0^c(\beta)$ at the field points β using equation (17) and the chosen values of S_r and mS_r. How the integration was performed will be discussed in Section A2.

(d) Generally, the calculated $S_0^c(\beta)$ will be different from the assumed function $S_0^a(\eta)$. Take as a new trial solution:

$$S_1^a(\eta)=[S_0^c(\beta)+S_0^a(\eta)]/2; \qquad \beta=\eta \tag{A1}$$

(e) Repeat the process until the calculated distribution is equal to the assumed distribution to the desired degree of accuracy, taken to be one part in 10^4 at all points in this work. Convergence took 10–15 iterations and was smooth for the values of m reported here.

Uniqueness was checked by starting with differenι values of $S_0^a(\eta)$. The final solution was independent of the assumed one within the limits of the parameters chosen.

A2. *Method of integration*

The integral in equation (15) is not easy to integrate directly because of the divergence at $\beta=\eta$. It can, however, be integrated analytically between any two points for constant $S(\eta)$ or $S(\eta)$ varying linearly. Consequently the interval 0–1 was broken up into 54 intervals over which, as an approximation, the stress was assumed to vary linearly. Integration then amounted to evaluating the sum:

$$I(\beta)\cong\sum_{1}^{54}\left\{\frac{2\sqrt{\beta}(S_i-S_{i+1})}{\sqrt{\eta_1}+\sqrt{\eta_{i+1}}}+\frac{S_i(\eta_{i+1}-\beta)+S_{i+1}(\beta-\eta_i)}{\eta_{i+1}-\eta_i}\times\ln\frac{(\sqrt{\beta}+\sqrt{\eta_{i+1}})|\sqrt{\beta}-\eta\sqrt{\eta_i}|}{(\sqrt{\beta}+\sqrt{\eta_i})|\sqrt{\beta}-\sqrt{\eta_{i+1}}|}\right\} \tag{A2}$$

It may be shown that this sum converges in the Cauchy principal value sense as $\beta\rightarrow\eta$. However, the sum cannot easily be evaluated for $\beta=\eta$. Since, for comparison of trial solutions and calculated solutions, it is necessary for the source point (η) to be equal to the field point (β), β was taken as $(\eta-10^{-6})$ to preclude division by zero, except for the points $\eta=0$ (and $\eta=0.01$) as discussed in Section A3. Using values β closer to η made no difference in the results as reported.

A3. *Treatment at the crack and craze tips*

As can be seen from equation (A2), the final term of the sum ($\eta=1$) diverges logarithmically as $\beta\rightarrow1$. The stress, however, is given by the log of $I(\beta)$ (see equation (17)), so that the stress has a doubly logarithmic divergence. The values reported are for $\beta=1-10^{-6}$. The handling of the situation at the craze tip is more complicated. As pointed out in the body of the text, the craze tip for our purposes is where yielding occurs. Actually, fibrillation occurs at some point beyond this, and by hypothesis this is the point at which craze fibrils are first seen. Now, as easily seen from equation (6) or (A2), at the craze tip ($\beta=0$), $\dot{w}=0$. At this point, nevertheless, the stress is the yield stress, and hence $S=1$, and the displacement is yielded material, but not craze fibrils.

In the calculations this situation was handled in two different ways.

(a) In this method, the stress was kept constant at a value of unity for $\eta=0$ and to $\eta=0.01$. These points did not enter into the iteration protocol described in Section A1. They do, however, enter into equation (A2), in controlling the displacement at all other points. This scheme was chosen to represent in a rough qualitative manner the portion of the stress–strain curve at yield. For the initial assumed stress distribution $S_0^a(\eta)$, the stress was decreased linearly from its value of unity at $\eta=0.01$ to the value of S_r at $\eta=0.02$, which was the same as at all other points to $\eta=1$. Iteration was carried out at the points 0.011, 0.015, 0.99, and all multiples of 0.02 up to $\eta=1$. The values of S at $\eta=0$ and $\eta=0.01$ were always kept at unity.

(b) In the second method, only the value of S at $\eta=0$ was kept equal to unity. For the starting distribution, S was decreased linearly to S_r at 0.02 and then handled as above. Iteration was carried out at all points except $\eta=0$, at which point S was always kept at unity.